LE MATEMATICHE
Vol. LII (1997) – Fasc. II, pp. 365–378

DECOMPOSITION OF THE BESSEL FUNCTIONS WITH RESPECT TO THE CYCLIC GROUP OF ORDER *n*

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Let n be an arbitrary positive integer. We decompose the functions

$$j_\nu(z) = \begin{cases} \Gamma(\nu+1)(\frac{z}{2})^{-\nu} J_\nu(z) & \text{if } z \neq 0 \\ 1 & \text{if } z = 0 \end{cases}, \qquad \nu \geq \frac{-1}{2},$$

where J_ν is the Bessel function of the first kind of order ν, as the sum of n functions $(j_\nu)_{[2n,2k]}$, $k = 0, 1, \dots, n-1$, defined by

$$(j_\nu)_{[2n,2k]}(z) = \frac{1}{n} \sum_{\ell=0}^{n-1} \exp(-\frac{2i\pi k\ell}{n}) j_\nu(z\exp(\frac{i\pi\ell}{n})), \quad z \in \mathbb{C}.$$

In this paper, we establish the close relation between these components and the hyper-Bessel functions introduced by Delerue [3]. The use of a technique described in an earlier work [1] leads us to derive, from the basic identities and relations for j_ν, other analogous for the components $(j_\nu)_{[2n,2k]}$ that turn out to be some integral representations of Sonine, Mehler and Poisson type, an operational representation and a differential equation of order $2n$. Thereafter, two identities for j_ν are expressed by the use of the components $(j_\nu)_{[2n,2k]}$.

Entrato in Redazione il 9 ottobre 1997.

1. Introduction.

All the notations and conventions begun in [1] will be continued in this paper. We recall in particular that $\Omega(I) \equiv \Omega$ denotes the space of complex functions admitting a Laurent expansion in an annulus I with center in the origin and for an arbitrary positive integer p, every function f in Ω can be written as the sum of p functions $f_{[p,r]}$; $r = 0, 1, \ldots, p-1$; defined by (cf. Ricci [14], p. 44, Eq.(3.3)):

$$f_{[p,r]}(z) = \frac{1}{p} \sum_{\ell=0}^{p-1} \omega_p^{-k\ell} f(\omega_p^{\ell} z), \qquad z \in I$$

with $\omega_p = \exp(\frac{2i\pi}{p})$ the complex p-root of unity.
Let n be an arbitrary positive integer, in view of (1.2) and (1.3) in [1], we have

$$\Omega = \bigoplus_{r=0}^{p-1} \Omega_{[p,r]} = \bigoplus_{\ell=0}^{pn-1} \Omega_{[pn,\ell]} = \bigoplus_{r=0}^{p-1} \bigoplus_{k=0}^{n-1} \Omega_{[pn,pk+r]}$$

from which we deduce that a function f in Ω can be written as the sum of pn functions $f_{[pn,\ell]}$; $\ell = 0, 1, \ldots, pn-1$; and if moreover, $f \in \Omega_{[p,r]}$, this decomposition coincides with the decomposition of f with respect to the cyclic group of order n. We have in fact,

$$f = \sum_{k=0}^{n-1} f_{[pn,pk+r]} \tag{1.1}$$

with

$$f_{[pn,pk+r]}(z) = \frac{1}{pn} \sum_{\ell=0}^{pn-1} \omega_{pn}^{-\ell(pk+r)} f(\omega_{pn}^{\ell} z) \tag{1.2}$$

or, equivalently,

$$f_{[pn,pk+r]}(z) = \frac{1}{n} \sum_{s=0}^{n-1} \omega_{pn}^{-s(pk+r)} f(\omega_{pn}^{s} z). \tag{1.3}$$

This paper deals with the decomposition with respect to the cyclic group of order n of one of the most important special functions, the function j_ν defined by

$$j_\nu(z) = \begin{cases} \Gamma(\nu+1)(\frac{z}{2})^{-\nu} J_\nu(z) & \text{if } z \neq 0 \\ 1 & \text{if } z = 0 \end{cases}, \qquad \nu \geq \frac{-1}{2}, \tag{1.4}$$

where J_ν is the Bessel function of the first kind of order ν.
Notice that $j_{-\frac{1}{2}}(z) = \cos z$.
The function j_ν belongs to $\Omega_{[2,0]}$ since we have

$$(1.5) \qquad j_\nu(z) = \sum_{m=0}^{\infty} \frac{\Gamma(\nu+1)}{\Gamma(m+\nu+1)\cdot m!}\cdot\left(\frac{-z^2}{4}\right)^m, \quad |z| < \infty$$

it follows then that we can write it as the sum of n functions $(j_\nu)_{[2n,2k]}$; $k = 0, 1, \ldots, n-1$; defined by

$$(1.6) \qquad (j_\nu)_{[2n,2k]}(z) = \frac{1}{n}\sum_{\ell=0}^{n-1} \omega_n^{-k\ell}\, j_\nu(\omega_{2n}^{\ell} z).$$

With the two additional parameters n and k the functions $(j_\nu)_{[2n,2k]}$ can be viewed as generalizations of the function j_ν since; for $n = 1$; we have $(j_\nu)_{[2,0]} = j_\nu$. Then we begin by situating the components $(j_\nu)_{[2n,2k]}$ among the generalizations in the literature of the function j_ν, more precisely, we shall state the relation between these components and the hyper-Bessel functions introduced by Delerue [3]. Thereafter, the use of the technique described in [1] leads us to derive, from the basic identities and relations for j_ν, other analogous for the components $(j_\nu)_{[2n,2k]}$ that turn out to be a hypergeometric serie representation, some integral representations of Sonine, Mehler and Poisson type, an operational representation and a differential equation of order $2n$. A Parseval formula and a nth-order circulant determinant will be also stated for the function j_ν.

2. Representation as a hypergeometric function.

We recall that the generalized hypergeometric function is defined by (see, for instance, Luke [12], p. 136, Eq. (1)):

$$(2.1) \qquad {}_pF_q(z) = {}_pF_q\left(\begin{matrix} a_1, & \ldots & , a_p \\ & & \\ b_1, & \ldots & , b_q \end{matrix}\ ;\ z\right) = \sum_{m=0}^{+\infty} \frac{(a_1)_m \cdots (a_p)_m}{(b_1)_m \cdots (b_q)_m}\cdot\frac{z^m}{m!}$$

where

- $(a)_m$ is the Pochlammer symbol defined by

$$(a)_m = \frac{\Gamma(a+m)}{\Gamma(a)}, \quad a \neq 0, -1, -2, \ldots ;$$

- p and q are positive integers or zero (interpreting an empty product as 1);
- z the complex variable;
- the numerator parameters a_i, $i = 1, \ldots, p$, and the denominator parameters b_j, $j = 1, \ldots, q$, take on complex values, providing that $b_j \neq 0, -1, -2, \ldots, j = 1, \ldots, q$.

The ${}_pF_q$ series in (2.1) converges for $|z| < \infty$ if $p \leq q$.
Now, to express $\left(j_\nu\right)_{[2n,2k]}$ by a generalized hypergeometric function, we start from:

$$j_\nu(z) = {}_0F_1\left(\begin{matrix} - \\ \nu+1 \end{matrix}\ ;\ \frac{-z^2}{4}\right) \tag{2.2}$$

which we deduce from (1.5) and (2.1). We write this expression under the form:

$$j_\nu(z) = S_{\frac{i}{2}}\big(\psi \circ g\big)(z)$$

where

- S_α, $\alpha \in \mathbb{C}$, is the scaling operator on Ω defined by $S_\alpha(f)(z) = f(\alpha z)$, for all $f \in \Omega$ and for all $z \in \mathbb{C}$;
- ψ and g the two functions given by

$$g(z) = z^2 \ \text{ and } \ \psi(z) = {}_0F_1\left(\begin{matrix} - \\ \nu+1 \end{matrix}\ ;\ z\right).$$

The use of Corollary II.3 and the identity (I-9) in [1] yield

$$\begin{aligned}\left(j_\nu\right)_{[2n,2k]}(z) = {} & \frac{1}{k!(\nu+1)_k}\left(\frac{iz}{2}\right)^{2k} \\ & {}_0F_{2n-1}\left(\begin{matrix} - & - \\ \Delta^*(n,k+1), & \Delta(n,\nu+1) \end{matrix}\ ;\ \left(\frac{iz}{2n}\right)^{2n}\right)\end{aligned} \tag{2.3}$$

where, for convenience, $\Delta(n,\alpha)$ (resp. $\Delta^*(n,k+1)$) stands for the set of n (resp. $n-1$) parameters $\frac{\alpha}{n}, \frac{\alpha+1}{n}, \ldots, \frac{\alpha+n-1}{n}$ (resp. $\Delta(n,k+1) \setminus \{\frac{n}{n}\}$).
Then an equivalent expression as an infinite series is deduced:

$$\left(j_\nu\right)_{[2n,2k]}(z) = \sum_{m=0}^{\infty} \frac{1}{(\nu+1)_{nm+k} \cdot (nm+k)!} \cdot \left(\frac{-z^2}{4}\right)^{nm+k} ; \quad |z| < \infty \tag{2.4}$$

or, equivalently,

$$\left(j_\nu\right)_{[2n,2k]}(z) = \frac{1}{k!(\nu+1)_k}\left(\frac{iz}{2}\right)^{2k} \cdot$$
$$\cdot \sum_{m=0}^{\infty} \frac{1}{\prod_{j=0}^{n-1}\left(\frac{k+1+j}{n}\right)_m \prod_{j=0}^{n-1}\left(\frac{\nu+1+j}{n}\right)_m} \left(\frac{iz}{2n}\right)^{2nm}.$$

Notice that the function $\left(j_{-\frac{1}{2}}\right)_{[2n,2k]}$ can be expressed by the trigonometric functions of order $2n$ and $2k$th kind defined by (cf. Erdélyi et al. [7], p. 215, Eq. (18)):

$$g_{n,k}(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(nm+k)!} \cdot z^{nm+k};\, n \in \mathbb{N}^*, \quad k = 0, 1, \ldots, n-1\,, \tag{2.5}$$

or, equivalently,

$$g_{n,k}(z) = \frac{(-z)^k}{k!}\, {}_0F_{n-1}\left(\begin{matrix} - \\ \Delta^*(n, k+1) \end{matrix} ;\ \left(\tfrac{-z}{n}\right)^n\right) \tag{2.6}$$

we have in fact,

$$\left(j_{-\frac{1}{2}}\right)_{[2n,2k]}(z) = (\cos)_{[2n,2k]}\,(z) = e^{\frac{ik\pi}{n}} \cdot g_{2n,2k}\left(e^{\frac{i\pi(n-1)}{2n}} z\right) \tag{2.7}$$

which we can be deduced from (2.3) and (2.6) since

$$\Delta^*(n, k+1) \cup \Delta(n, \frac{1}{2}) \equiv\ \Delta^*(2n, 2k+1).$$

Also, among the consequences of the identity (2.3), we mention the possibility of stating a relation between ${}_0F_{2n-1}$ and Bessel functions that generalizes the Carlson ones [2]. Indeed, if we combine (2.3) and (1.6) we obtain

$$(2.8)\quad {}_0F_{2n-1}\left(\begin{matrix} - & - \\ \Delta^*(n, k+1) & \Delta(n, \nu+1+k) \end{matrix} ;\ z\right) =$$
$$= (-1)^k \frac{\Gamma(\nu+1+k)k!}{n} \left(\frac{\xi}{2}\right)^{-\nu-2k} \sum_{h=0}^{n-1} \exp\left[\frac{i\pi h(\nu+2k)}{n}\right] J_\nu\left(\xi e^{\frac{i\pi h}{n}}\right),$$

where $\xi = 2niz^{\frac{1}{2n}}$.
Two special cases of this identity are worthy of note here:
If we set $n = 2$, $k = 0$ and $c = \frac{\nu+1}{2}$ (or $n = 2$, $k = 1$ and $c = \frac{\nu+2}{2}$) in (2.8) and we use the well known identity (cf. [15], p. 203):

$$I_\nu(z) = e^{-\frac{1}{2}\nu\pi i} J_\nu\big(ze^{i\frac{\pi}{2}}\big)$$

we derive the following relations stated by Carlson (cf. [2], p. 233, Eq. (7)):

$$\text{(2.9)} \qquad \frac{1}{\Gamma(2c)}\, {}_0F_3\left(\begin{matrix} - \\ \frac{1}{2}, \quad c, \quad c+\frac{1}{2}\end{matrix} ; z\right) = \\ = \frac{1}{2}\big(2z^{\frac{1}{4}}\big)^{1-2c}\big[I_{2c-1}\big(4z^{\frac{1}{4}}\big) + J_{2c-1}\big(4z^{\frac{1}{4}}\big)\big]$$

$$\text{(2.10)} \qquad \frac{1}{\Gamma(2c)}\, {}_0F_3\left(\begin{matrix} - \\ \frac{3}{2}, \quad c, \quad c+\frac{1}{2}\end{matrix} ; z\right) = \\ = \frac{1}{2}\big(2z^{\frac{1}{4}}\big)^{-2c}\big[I_{2c-2}\big(4z^{\frac{1}{4}}\big) - J_{2c-2}\big(4z^{\frac{1}{4}}\big)\big].$$

3. Hyper-Bessel functions.

P. Delerue [3] generalized the Bessel functions J_ν by replacing the index ν by n parameters $\nu_1, \nu_2, \ldots, \nu_n$ that is:

$$\text{(3.1)} \qquad J^{(n)}_{\nu_1,\nu_2,\ldots,\nu_n}(z) = \frac{\big(\frac{1}{n+1}z\big)^{\sum \nu_i}}{\prod \Gamma(\nu_i+1)}\, {}_0F_n\left(\begin{matrix} - \\ (\nu_i+1)\end{matrix} ; -\big(\tfrac{z}{n+1}\big)^{n+1}\right)$$

which he called hyper-Bessel functions of order n and of index $\nu_1, \nu_2, \ldots\ldots \nu_n$. The same generalization was obtained thirty years after by Klyuchantsev [10] where the functions (3.1) were called Bessel functions of vector index and designated by $J_{(\nu_1,\nu_2,\ldots,\nu_n)}$.
For convenience, we set

$$\text{(3.2)} \qquad j_{(\nu_1,\nu_2,\ldots,\nu_{r-1})}(z) = \prod \Gamma(\nu_i+1)\Big(\frac{z}{r}\Big)^{-\sum \nu_i} J_{(\nu_1,\nu_2,\ldots,\nu_{r-1})}(z) = \\ = {}_0F_{r-1}\left(\begin{matrix} - \\ (\nu_i+1)\end{matrix} ; -\big(\tfrac{z}{r}\big)^r\right), \quad r \in \mathbb{N}^*,$$

where the summation $\sum \nu_j$ and the multiplication $\prod \Gamma(\nu_j + 1)$ are carried out over all j from 1 to $r-1$ and for the sake of brevity (a_j) stands for the sequence of $(r-1)$ parameters $a_1, a_2, \ldots, a_{r-1}$.

Next, we purpose to express $\big(j_\nu\big)_{[2n,2k]}$ through the functions designated by (3.2).
From the three parameters ν, n and k, one defines a vector $\boldsymbol{v} \in \mathbb{R}^{2n-1}$ as follows:

$$\boldsymbol{v}(n,k,\nu) \equiv \boldsymbol{v}\big(\nu_i(n,k,\nu)\big)_{1\le i\le 2n-1}$$

where the $(n-1)$ first components ν_i are given by $\Delta^*(n, k+1)$ and the n other by the set $\Delta(n, \nu+k)$.
Here, we introduce, for notational convenience, the vector $\mathbf{1}_{2n-1}$ in $\mathbb{R}^{2n-1}$ having all components equal to unity.
From (2.3) and (3.2) we deduce:

$$\big(j_\nu\big)_{[2n,2k]}(z) = \frac{(-1)^k}{k!(\nu)_k}\Big(\frac{z}{2}\Big)^{2k} j_{\boldsymbol{v}(n,k,\nu)-\mathbf{1}_{2n-1}}\big(ize^{\frac{i\pi}{2n}}\big). \tag{3.3}$$

For $\nu = \frac{1}{2}$, we have

$$g_{2n,2k}(z) = \frac{z^{2k}}{(2k)!} j_{\boldsymbol{v}(n,k,\frac{1}{2})-\mathbf{1}_{2n-1}}(z) \tag{3.4}$$

which reduces, for $n = 1$, to the well known identity:

$$\cos z = j_{-\frac{1}{2}}(z).$$

4. Integral representations.

We recall that the Bessel functions have the following integral representation known as Sonine integral (see, for instance, [15], p. 373, Eq. (1) or [6], Vol.II, p. 194, Eq. (63))

$$J_{\nu+\mu}(ay) = \frac{a^\mu y^{-\mu-\nu}}{2^{\mu-1}\Gamma(\mu)} \int_0^y (y^2 - x^2)^{\mu-1} x^{\nu+1} J_\nu(ax)\,dx, \tag{4.1}$$

$$\operatorname{Re}\nu > -1 \text{ and } \operatorname{Re}\mu > 0$$

which reduces, for $\nu = -\frac{1}{2}$, to so-called Mehler representation (see, for example, [11], p. 114, Eq. (5.10.3) or [6], Vol.II, p. 190, Eq. (34)):

$$J_{\mu-\frac{1}{2}}(ay) = \frac{a^{\mu-\frac{1}{2}} y^{\frac{1}{2}-\mu}}{2^{\mu-\frac{3}{2}}\Gamma(\pi)\Gamma(\mu)} \int_0^y (y^2 - x^2)^{\mu-1} \cos(ax)\,dx, \quad \operatorname{Re}\mu > 0. \tag{4.2}$$

Using (1.4) and a change of variable, one obtains :

$$j_{\nu+\mu}(ax) = \frac{2\Gamma(\mu+\nu+1)}{\Gamma(\mu)\Gamma(\nu+1)} \int_0^1 (1-t^2)^{\mu-1} t^{2\nu+1} j_\nu(atx)\, dt, \tag{4.3}$$

$$\operatorname{Re}\nu > -1 \text{ and } \operatorname{Re}\mu > 0$$

$$j_\nu(ax) = \frac{2\Gamma(\nu+1)}{\Gamma(\frac{1}{2})\Gamma(\nu+\frac{1}{2})} \int_0^1 (1-t^2)^{\nu-\frac{1}{2}} \cos(atx)\, dt, \quad \operatorname{Re}\nu > -\frac{1}{2}\, . \tag{4.4}$$

If we apply the projection operator $\Pi_{[2n,2k]}$ to each member of these two formulas considered as functions of the variable x and we use the integral representation (IV-2) in [1], we obtain the following proposition:

Proposition. *The functions $\left(j_\nu\right)_{[2n,2k]}$ have:*

i) a Mehler type integral representation

$$\left(j_\nu\right)_{[2n,2k]}(xe^{\frac{(1-n)i\pi}{2n}}) = \tag{4.5}$$
$$= \frac{2\Gamma(\nu+1)}{\Gamma(\frac{1}{2})\Gamma(\nu+\frac{1}{2})} \int_0^1 (1-t^2)^{\nu-\frac{1}{2}} g_{2n,2k}(xt)\, dt, \quad \operatorname{Re}\nu > -\frac{1}{2}$$

ii) a Sonine type integral representation:

$$\left(j_{\nu+\mu}\right)_{[2n,2k]}(ax) = \tag{4.6}$$
$$= \frac{2\Gamma(\mu+\nu+1)}{\Gamma(\mu)\Gamma(\nu+1)} \int_0^1 (1-t^2)^{\mu-1} t^{2\nu+1} \left(j_\nu\right)_{[2n,2k]}(atx)\, dt,$$

$$\operatorname{Re}\nu > -1 \ \textit{and} \ \operatorname{Re}\mu > 0$$

iii) a Poisson type integral representation:

$$\left(j_\nu\right)_{[2n,2k]}(re^{i\theta}) = \tag{4.7}$$
$$= \int_0^{2\pi} P_{n,k}(R, r, \phi-\theta) j_\nu(Re^{i\phi})\, d\phi, \quad r < R, \ 0 \le \theta \le 2\pi$$

or, equivalently,

$$(4.8)\qquad \big(j_\nu\big)_{[2n,2k]}(re^{i\theta}) =$$
$$= \int_0^{2\pi} P_{n,k}(R,r,\phi-\theta)\big(j_\nu\big)_{[2n,2k]}(Re^{i\phi})\,d\phi,\quad r < R,\ 0 \le \theta \le 2\pi$$

where the kernel $P_{n,k}(R,r,\phi-\theta)$ *is defined by*

$$P_{n,k}(R,r,\phi-\theta) =$$
$$= \frac{(R^{2(n-k)} - r^{2(n-k)})R^k r^k e^{-ik(\phi-\theta)} + (R^{2k} - r^{2k})R^{n-k} r^{n-k} e^{i(n-k)(\phi-\theta)}}{2\pi\big(R^{2n} + r^{2n} - 2R^n r^n \cos n(\phi-\theta)\big)}\,.$$

A particular case of (4.5) corresponding to $k = 0$ can be written using (3.3) as follows:

$$(4.9)\qquad j_{(-\frac{1}{n},-\frac{2}{n},\dots,-\frac{(n-1)}{n},\frac{\nu}{n},\frac{\nu-1}{n},\dots,\frac{\nu-(n-1)}{n})}(x) =$$
$$= \frac{2\Gamma(\nu+1)}{\Gamma(\frac{1}{2})\Gamma(\nu+\frac{1}{2})}\int_0^1 (1-t^2)^{\nu-\frac{1}{2}} g_{2n,0}(xt)\,dt,\quad \mathrm{Re}\,\nu > -\frac{1}{2}\,.$$

This identity is also a particular case of an interesting integral representation given by Dimovski and Kiryakova (cf.[4], p. 32, Eq. (15) or [9], p. 34, Eq. (8)):

$$(4.10)\qquad j_{(\nu_1,\dots,\nu_q)}(x) = \sqrt{\frac{q+1}{(2\pi)^q}}\prod_{\ell=1}^{q}\Gamma(\nu_\ell+1)\cdot$$
$$\cdot\int_0^1 G_{q\,q}^{q\,0}\left(t\,\middle|\,\begin{matrix}(\nu_k)\\ (\frac{k}{q+1}-1)\end{matrix}\right) g_{q+1,0}(xt^{\frac{1}{q+1}})\,dt,$$

where $G_{p\,q}^{m\,n}\left(z\,\middle|\,\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right)$ designates the Meijer's G-function (see, for instance, [5], Vol.I, p. 207, [12], p. 143 or [13], p. 2 for the definition).
To verify that (4.9) is a special case of (4.10), one can use the identities (2), (4) and (5), p. 150 in the book [12] and the formula

$$G_{1\,1}^{1\,0}\left(x\,\middle|\,\begin{matrix}\alpha+\beta+1\\ \alpha\end{matrix}\right) = \frac{x^\alpha(1-x)^\beta}{\Gamma(\beta+1)},\quad 0 < x < 1 \qquad \text{(cf. [13], p. 37).}$$

5. An operational representation.

We recall that the Bessel functions J_ν satisfy the following identity (cf. [15], p. 46, Eq. (6)):

$$(-1)^m z^{-\nu-m} J_{\nu+m}(z) = \Big(\frac{d}{zdz}\Big)^m \big(z^{-\nu} J_\nu(z)\big), \quad m \in \mathbb{N}$$

so, for j_ν, we have

$$j_{\nu+m}(z) = (-2)^m \frac{\Gamma(\nu+m+1)}{\Gamma(\nu+1)} \Big(\frac{d}{zdz}\Big)^m \Big(j_\nu(z)\Big), \quad m \in \mathbb{N}. \tag{5.1}$$

According to the decomposition

$$\Omega = \bigoplus_{\ell=0}^{2n-1} \Omega_{[2n,\ell]}$$

the differential operator $\big(\frac{d}{zdz}\big)^m$ is homogeneous of degree $2n-2m$. So, by virtue of Theorem III-1 in [1], we have

$$\Pi_{[2n,2k]} \circ \Big(\frac{d}{zdz}\Big)^m = \Big(\frac{d}{zdz}\Big)^m \circ \Pi_{[2n,\widehat{2k+2m}]}$$

which leads us, if we apply the projection operator $\Pi_{[2n,2k]}$ to the two members of (5.1), to obtain

$$\big(j_{\nu+m}\big)_{[2n,2k]}(z) = (-2)^m \frac{\Gamma(\nu+m+1)}{\Gamma(\nu+1)} \Big(\frac{d}{zdz}\Big)^m \Big(\big(j_\nu\big)_{[2n,\widehat{2k+2m}]}(z)\Big), \ m \in \mathbb{N}.$$

In particular, if m is a multiple of n, that is $m = nr$, we have:

$$\big(j_{\nu+nr}\big)_{[2n,2k]}(z) = (-2)^{nr} \frac{\Gamma(\nu+nr+1)}{\Gamma(\nu+1)} \Big(\frac{d}{zdz}\Big)^{nr} \Big(\big(j_\nu\big)_{[2n,2k]}(z)\Big).$$

6. A differential equation.

We purpose in this section to establish a differential equation satisfied by the components $\left(j_\nu\right)_{[2n,2k]}$.

We recall that the functions $z \to j_\nu(\lambda z)$ are solutions of the differential equation:

$$B_2 u = -\lambda^2 u \tag{6.1}$$

where $B_2 = B_2(\nu) = D_z^2 + \frac{2\nu+1}{z} D_z$, $D_z = \frac{d}{dz}$ is the classical Bessel differential operator, with the initial conditions:

$$u(0) = 1, \quad u'(0) = 0.$$

The action of B_2 on both sides of (6.1) and the use of (6.1) to eliminate B_2 in the right side yield a differential equation of order four satisfied by the functions $z \to j_\nu(\lambda z)$. The reiteration of this process $(r-1)$ times gives rise to the following differential equation:

$$B_2^r\left(j_\nu(\lambda z)\right) = \left(-\lambda^2\right)^r j_\nu(\lambda z). \tag{6.2}$$

The action of the projection operators $\prod_{[2n,2k]}$ on both sides of (6.2), with $n = r$, gives us, in view of Theorem III-1 in [1] since B_2^n is homogeneous of degree zero, the following system satisfied by the functions $z \to \left(j_\nu\right)_{[2n,2k]}(\lambda z)$:

$$\left(\textstyle\sum_{n,k}(\nu)\right) \qquad \begin{cases} B_2^n u(z) = (-\lambda^2)^n u(z), & \lambda \in \mathbb{C} \\ \dfrac{d^\ell u}{dz^\ell}(0) = \delta_{2k\ell} c_k \cdot \lambda^\ell, & \ell \in \{0, 1, \ldots, 2n-1\} \end{cases}$$

where δ_{ij} is the Kronocker symbol and the constants c_k are given by

$$c_k = \frac{(-1)^k (2k)!}{2^{2k} k! (\nu+1)_k}$$

which we can be deduced from (2.4).

We observe that

i) For $k = 0$ and $z \in]0, +\infty[$ the system $\left(\sum_{n,k}(\nu)\right)$ coincides with a class of the initial value problem for singular differential equation containing an operator of the form

$$B_r = \frac{d^r}{dz^r} + \frac{b_1}{z}\frac{d^{r-1}}{dz^{r-1}} + \cdots + \frac{b_{r-1}}{z^{r-1}}\frac{d}{dz}$$

with coefficients $b_i = b_i(\nu_1, \ldots, \nu_{r-1})$ depending on parameters $\nu_1, \ldots, \nu_{r-1}$ considered by many authors, we quote, for instance, Delerue [3], Dimovski-Kiryakova [4], Kiryakova [9] and Klyuchantsev [10].

ii) For $\nu = -\frac{1}{2}$, the solutions of $\left(\sum_{n,k}(\nu)\right)$ reduce to trigonometric functions of order $2n$ and $2k$th kind $z \to g_{2n,2k}(\lambda z)$ defined by (2.5) (See for instance Erdélyi et al. [7], p. 215, Eqs. (19) and (20)).

iii) For $n = 2$ and $k = 0$, we have

$$j_{(-\frac{1}{2}, \frac{\nu}{2}, \frac{\nu-1}{2})}(\lambda z) = {}_0F_3 \left(\begin{matrix} & - & & ; -\left(\frac{\lambda z}{4}\right)^4 \\ -\frac{1}{2}, & \frac{\nu}{2}, & \frac{\nu-1}{2} & \end{matrix} \right)$$

is solution of the system:

$$\begin{cases} B_2^2 u = D_z^4 u + \frac{2(2\nu+1)}{z} D_z^3 u + \frac{4\nu^2-1}{z^2} D_z^2 u + \frac{1-4\nu^2}{z^3} D_z u = -\lambda^4 u \\ u(0) = 1, u^{(\ell)}(0) = 0; \quad \ell \in \{1, 2, 3\} \end{cases}$$

which we can verify from the identities (1.40), (1.41) and (1.42), when $r = 4$, p. 359 in Klyuchantsev's paper [10].

Remark. The integral representation (4.5) can be used to define a transmutation operator between $\dfrac{d^{2n}}{dz^{2n}}$ and $B_2^n(\nu)$ just as (4.6) can be used to define a transmutation operator between $B_2^n(\nu)$ and $B_2^n(\nu + \mu)$.

7. A Parseval formula.

A similar proof of Proposition V-1 in [1] can be used here to state the following

Proposition 7.1. *Let $(f, g) \in \left(\Omega_{[p,0]}\right)^2$ and $(x, y) \in I^2$ we have:*

$$\sum_{k=0}^{n-1} f_{[pn,pk]}(x) \cdot \overline{g_{[pn,pk]}}(y) = \frac{1}{n} \sum_{\ell=0}^{n-1} f\left(\omega_{pn}^{\ell} x\right) \cdot \bar{g}\left(\omega_{pn}^{\ell} y\right).$$

The special case, where $f = g$ and $x = y$, amounts to the following Parseval formula:

$$\sum_{k=0}^{n-1} \left| f_{[pn,pk]}(x) \right|^2 = \frac{1}{n} \sum_{\ell=0}^{n-1} \left| f\left(\omega_{pn}^{\ell} x\right) \right|^2.$$

From which we deduce a Parseval formula for the function j_ν:

$$\sum_{k=0}^{n-1} \left| (j_\nu)_{[2n,2k]}(z) \right|^2 = \frac{1}{n} \sum_{\ell=0}^{n-1} \left| j_\nu\left(\omega_{2n}^{\ell} z\right) \right|^2 .$$

8. A nth-order circulant determinant.

We recall that the nth-order circulant determinant is (cf.[8], p. 1112)

$$\begin{vmatrix} x_0 & x_{n-1} & \cdots & x_1 \\ x_1 & x_0 & \cdots & x_2 \\ \vdots & \vdots & \ddots & \vdots \\ x_{n-1} & x_{n-2} & \cdots & x_0 \end{vmatrix} = \prod_{\ell=0}^{n-1} \left(\sum_{k=0}^{n-1} \omega_n^{k\ell} x_k \right)$$

from which we deduce that any function $f \in \Omega_{[2,0]}$ satisfies the following identity:

$$\prod_{\ell=0}^{n-1} f(\omega_{2n}^{\ell} z) = \begin{vmatrix} f_{[2n,0]}(z) & f_{[2n,2n-2]}(z) & \cdots & f_{[2n,2]}(z) \\ f_{[2n,2]}(z) & f_{[2n,0]}(z) & \cdots & f_{[2n,4]}(z) \\ \vdots & \vdots & \ddots & \vdots \\ f_{[2n,2n-2]}(z) & f_{[2n,2n-4]}(z) & \cdots & f_{[2n,0]}(z) \end{vmatrix} \tag{8.1}$$

since we have

$$f\left(\omega_{2n}^{\ell} z\right) = \sum_{k=0}^{n-1} f_{[2n,2k]}\left(\omega_{2n}^{\ell} z\right) = \sum_{k=0}^{n-1} \omega_n^{k\ell} f_{[2n,2k]}(z).$$

Now, if we set $f = j_\nu$ in (8.1), we obtain the nth-order circulant determinant:

$$\prod_{\ell=0}^{n-1} j_\nu(\omega_{2n}^{\ell} z) = \begin{vmatrix} (j_\nu)_{[2n,0]}(z) & (j_\nu)_{[2n,2n-2]}(z) & \cdots & (j_\nu)_{[2n,2]}(z) \\ (j_\nu)_{[2n,2]}(z) & (j_\nu)_{[2n,0]}(z) & \cdots & (j_\nu)_{[2n,4]}(z) \\ \vdots & \vdots & \ddots & \vdots \\ (j_\nu)_{[2n,2n-2]}(z) & (j_\nu)_{[2n,2n-4]}(z) & \cdots & (j_\nu)_{[2n,0]}(z) \end{vmatrix}$$

which reduces, for $n = 2$, to

$$j_\nu(z) j_\nu(iz) = (j_\nu)^2_{[4,0]}(z) - (j_\nu)^2_{[4,2]}(z).$$

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